KU LEUVEN


# Short Solutions to Nonlinear Systems of Equations

September 2017

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# Question

880 6 8

## Why do the signature and ciphertext size increase in post-quantum schemes?

Why do the signature and ciphertext size increase in post-quantum schemes? And is there any table or comparison as to how much this increase happens for each scheme type (lattice, multivariate, code-based, etc.)

`encryption` `signature` `post-quantum-cryptography`

asked Jan 5 at

typos
**354**

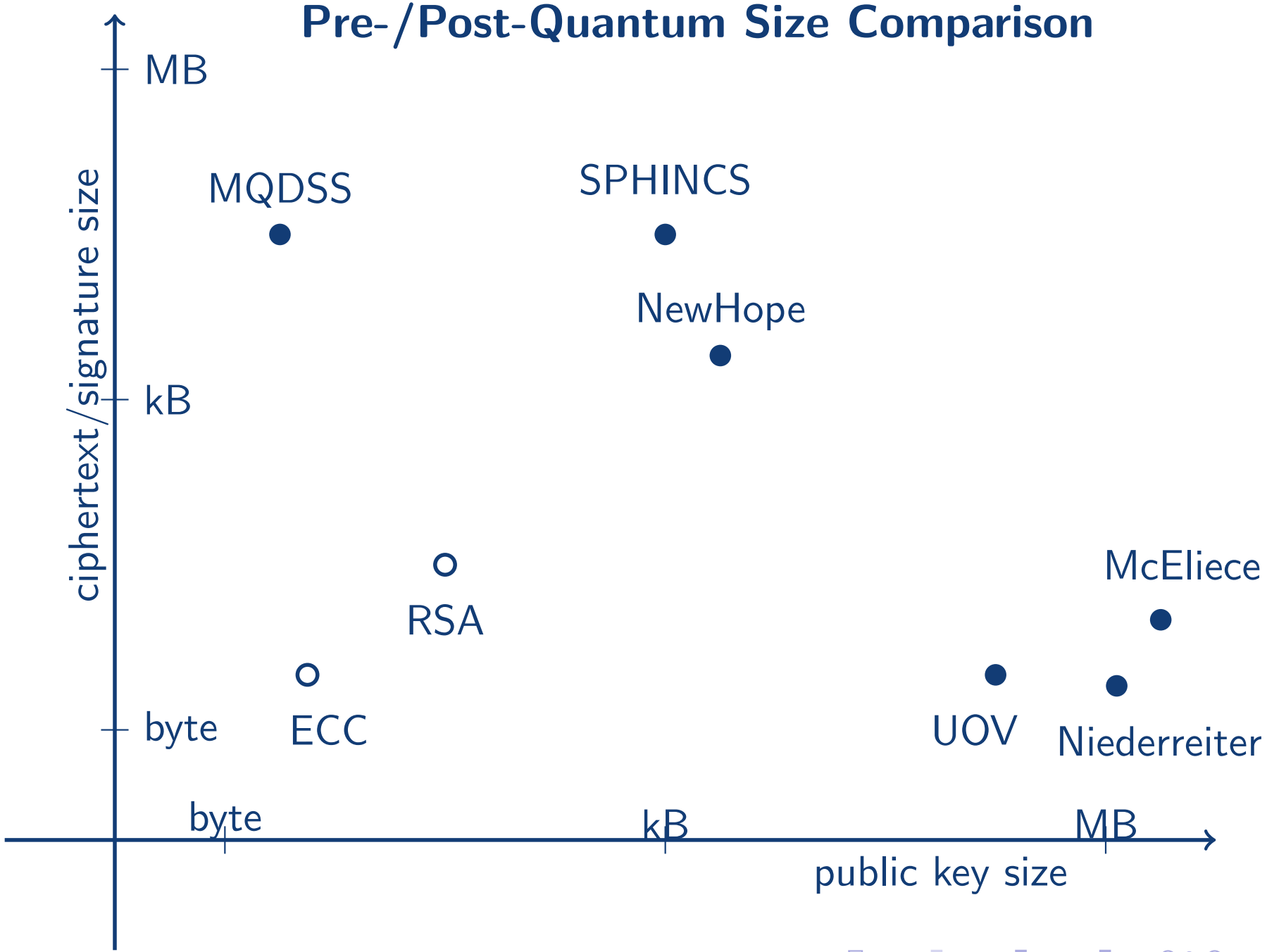
I'm really **not sure at all** about what I'm going to say, but I've heard about the so-called *Grover's quantum algorithm*, which can improve the efficiency of classical algorithms, and therefore improve classical attacks. This is a good forum to read about it: pqcrypto.org/pqc-forum/20161028-152555.txt – Daniel Jan 5 at 21:41

Increase compared to what? The message? Asymmetric schemes? I guess the best we can do is list signature and ciphertext sizes for specific strengths and a minimum amount of message size, say 256 bits for encryption and 512 bits for signature generation (enough for most hybrid systems). – Maarten Bodewes Jan 5 at 22:40

@SolidSnake That's why we can just increase the key size for symmetric cryptosystems. But, for public key crypto, Shor's algorithm breaks them in polynomial time, so people design new alternatives, but they usually have large signature size. – typos Jan 5 at 23:20

I know about that, what I do not know (like Maarten) is what are you comparing the sizes with? MPKC cryptosystems do not have that large ciphertext sizes in general – Daniel Jan 5 at 23:27

# Pre-/Post-Quantum Size Comparison



# Hard Problems

| scheme | type | hard problem | scales with |
|---|---|---|---|
| RSA/ECC | number theory | dlog/factoring | $\log_2 n \sim \kappa^c,\ \log_2 p \sim \kappa$ |
| Niederreiter | coding theory | syndrome decoding | $tm \times n \sim \kappa^2 \log_2 \kappa$ |
| UOV | MQ | MQ problem | $m \times n(n+1)/2 \sim \kappa^3$ |
| MQDSS | zero-knowledge | hiding/binding commit | $\kappa^2$ |
| NewHope | lattice | (R)LWE | $\dim(\mathcal{L}) \times \log_2 q \sim \kappa^c$ |
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$\Longrightarrow$ linear in search space $\sim q^n$

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- kills algebraic attack using Gröbner basis algorithms (Buchberger/F4/XL)
- infeasible to list all solutions

$$\deg(\mathcal{P}) > 1$$

- otherwise it is a linear algebra problem with short solutions
- quickly solved with lattice reduction

# Design Principle 1

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# Design Principle 1

$$\log_2 \beta \geq \kappa$$

- kills Arora-Ge-type algebraic attack
  - add equations $\prod_{j=\lfloor -\beta \rfloor}^{\lceil \beta \rceil} (x_i - j) = 0$
- kills bit-expansion algebraic attack
  - treat bits in bit expansion of $x_0, \ldots, x_{n-1}$ as variables
  - add equations $x_i^{(j)}(1 - x_i^{(j)}) = 0$
  - add equations $x_i^{(j+1)} - 2x_i^{(j)} = 0$
  - add equations $x_i - \sum 2^j x_i^{(j)} = 0$

# Design Principle 2

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- makes guess+algebra infeasible
  - guess small variables until system is determined
  - solve using Gröbner basis algorithm
  - BUT: random guess destroys short solution wop.

# Design Principle 3

$$\log_2 \|\mathbf{x}\|^2/n \geq \log_2 q$$

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- kills Coppersmith-type lattice attacks

$$\mathcal{P}(x_0, x_1) = 0 \quad \Leftrightarrow \quad \left( \begin{array}{|c|} \hline \\ \\ \hline \end{array} \right) \begin{pmatrix} x_0^2 \\ x_0 x_1 \\ x_1^2 \\ x_0 \\ x_1 \\ 1 \end{pmatrix} = 0$$

# Design Principle 4

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  - ignore higher order monomials;
  - just reduce submatrix associated with linear monomials
  - hope for the best

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- makes restricted lattice-reduction infeasible
  - ignore higher order monomials;
  - just reduce submatrix associated with linear monomials
  - hope for the best
- and also naïve guessing

# Design Principle 5

- generalized shortness requirement: $\mathbf{x}^{\mathsf{T}} W \mathbf{x} \leq \beta^2$ for some $W \in \mathbb{Z}^{n \times n}$

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- $\mathrm{rank}(W + W^{\mathsf{T}}) \approx \#$ of variables that must be short
- $\dim(\mathrm{var}(\mathcal{P})) \approx \#$ degrees of freedom in choosing solutions

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  - use LLL to find oil variables such that
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    $\sum_{i=n-o}^{n} (S^{-1})_{[:,i]} x_i$ is short ($\sim q^{(n-o+m)/(n+1)}$)
  - solution: $\underbrace{\sum_{i=0}^{n-o-1} (S^{-1})_{[:,i]} x_i}_{\text{vinegar contribution}} + \underbrace{\sum_{i=n-o}^{n} (S^{-1})_{[:,i]} x_i}_{\text{oil contribution}}$

# UOV Signature Scheme

- public key: $\mathcal{P} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ (multivariate quadratic)
- secret key: $(\mathcal{F}, S)$ such that $\mathcal{P} = \mathcal{F} \circ S$ and such that $\mathcal{F}$ is *linear* in the *oil* variables $x_{n-o}, \ldots, x_{n-1}$

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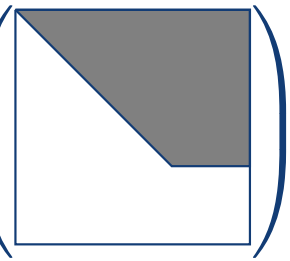

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- signature generation:
  - choose random *vinegar* variables $x_0, \ldots, x_{n-o-1} \xleftarrow{\$} \mathbb{F}_q$
  - $\mathcal{F}(\mathbf{x}) = \mathsf{H}(d)$ becomes *linear*, so solve
  - $\mathbf{s} \leftarrow S^{-1}\mathbf{x}$

# UOV Reconciliation Attack

- UOV Reconciliation Attack: *find* $S^{-1}$ *from* $\mathcal{P}$
  - treat coefficients of $S^{-1}$ as indeterminates
  - compute $\mathcal{F} = \mathcal{P} \circ S^{-1}$ symbolically
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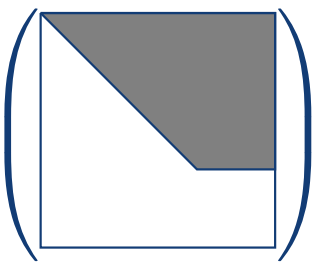

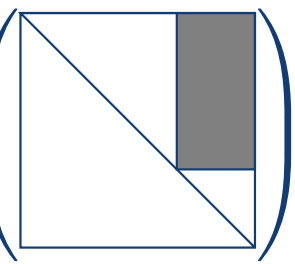

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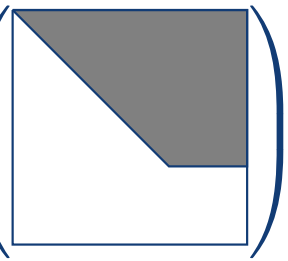

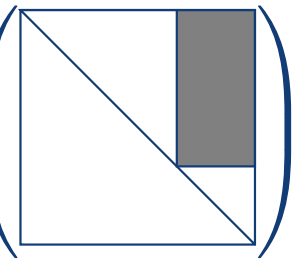

$$\#\text{eqns} = mo(o+1)/2 \qquad \#\text{vars} = (n-o)o$$

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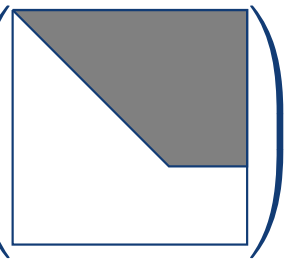

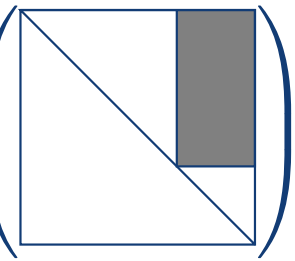

$$\#\text{eqns} = mo(o+1)/2 \qquad \#\text{vars} = (n-o)o$$

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# UOV Reconciliation Attack

- UOV Reconciliation Attack: *find* $S^{-1}$ *from* $\mathcal{P}$
  - treat coefficients of $S^{-1}$ as indeterminates
  - compute $\mathcal{F} = \mathcal{P} \circ S^{-1}$ symbolically
  - equate oil-times-oil coefficients to zero
  - solve system of quadratic equations

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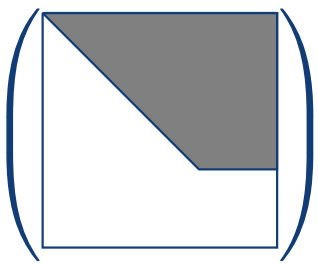

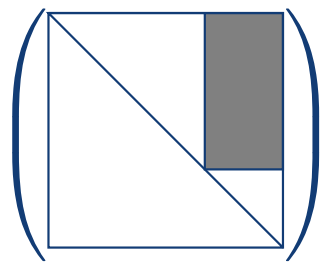

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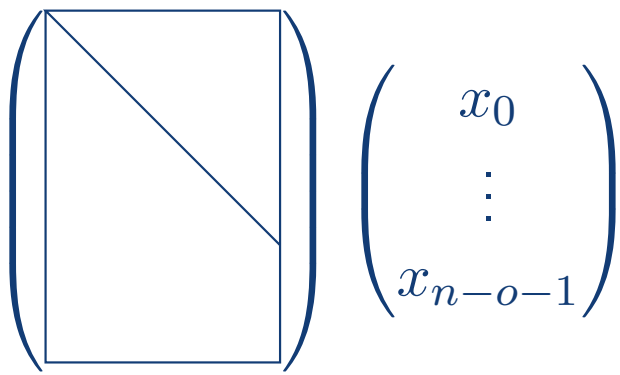

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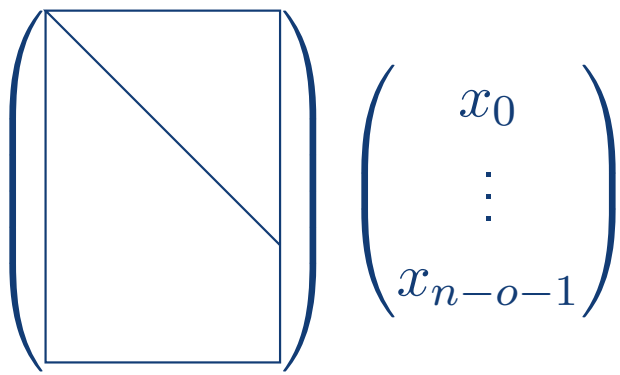

- small vinegar variables guarantee a short vinegar contribution

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- use LLL to find short vector in lattice spanned by

$$\begin{pmatrix} (S^{-}1)_{[:,-]}\mathbf{x}^{(p)} \\ 1 \end{pmatrix}, \begin{pmatrix} (S^{-1}_{[:,-]}\mathbf{x}_0^{(k)} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} (S^{-1})_{[:,-]}\mathbf{x}_2^{(k)} \\ 0 \end{pmatrix}$$

# Map of Public Key Crypto

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hard problem

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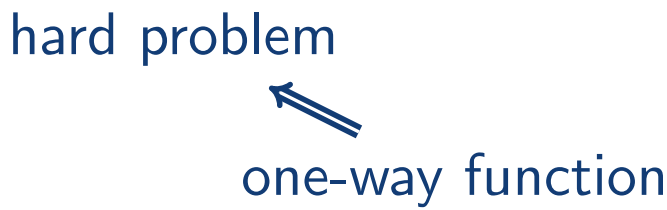

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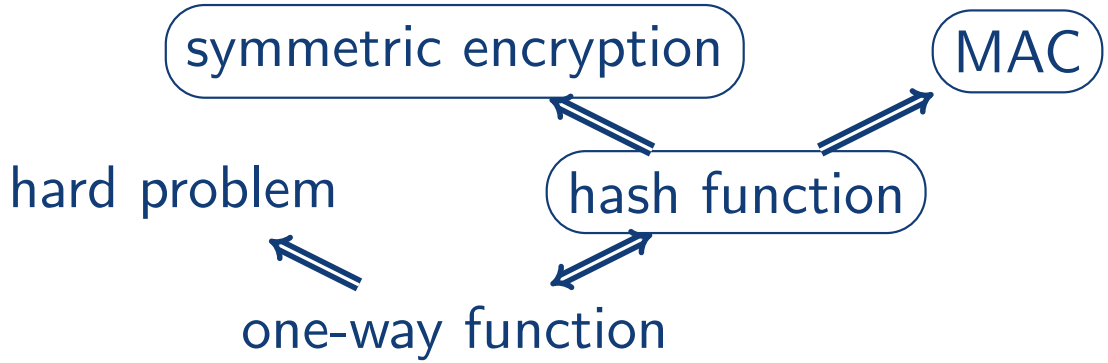

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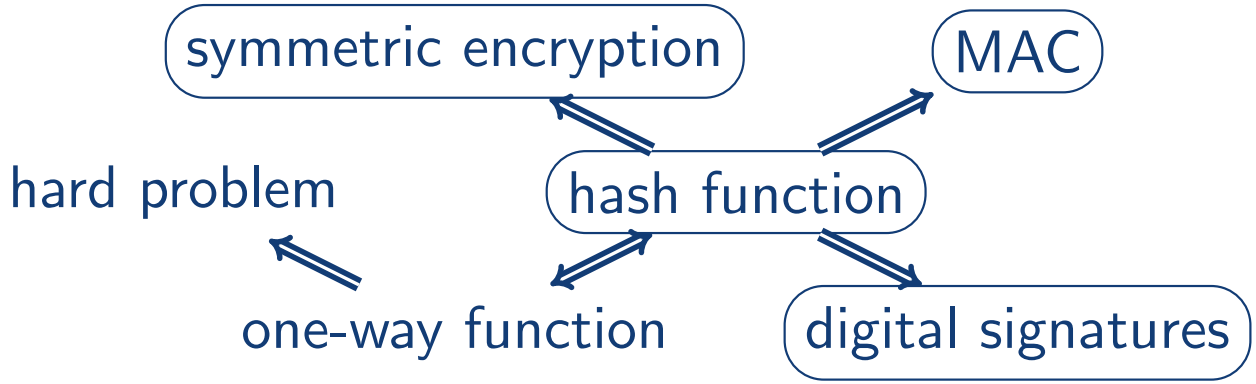

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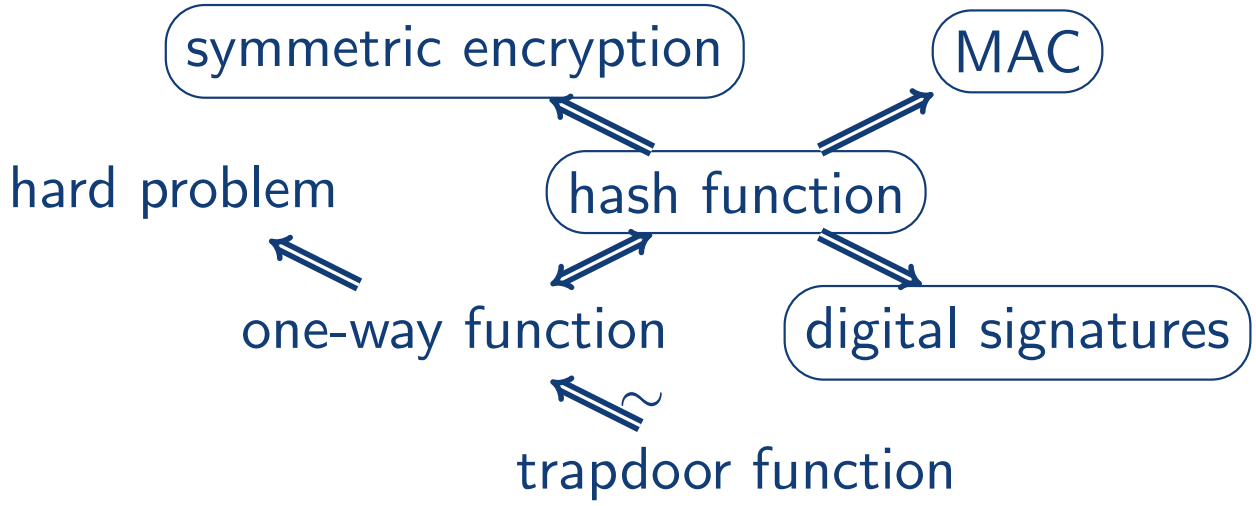

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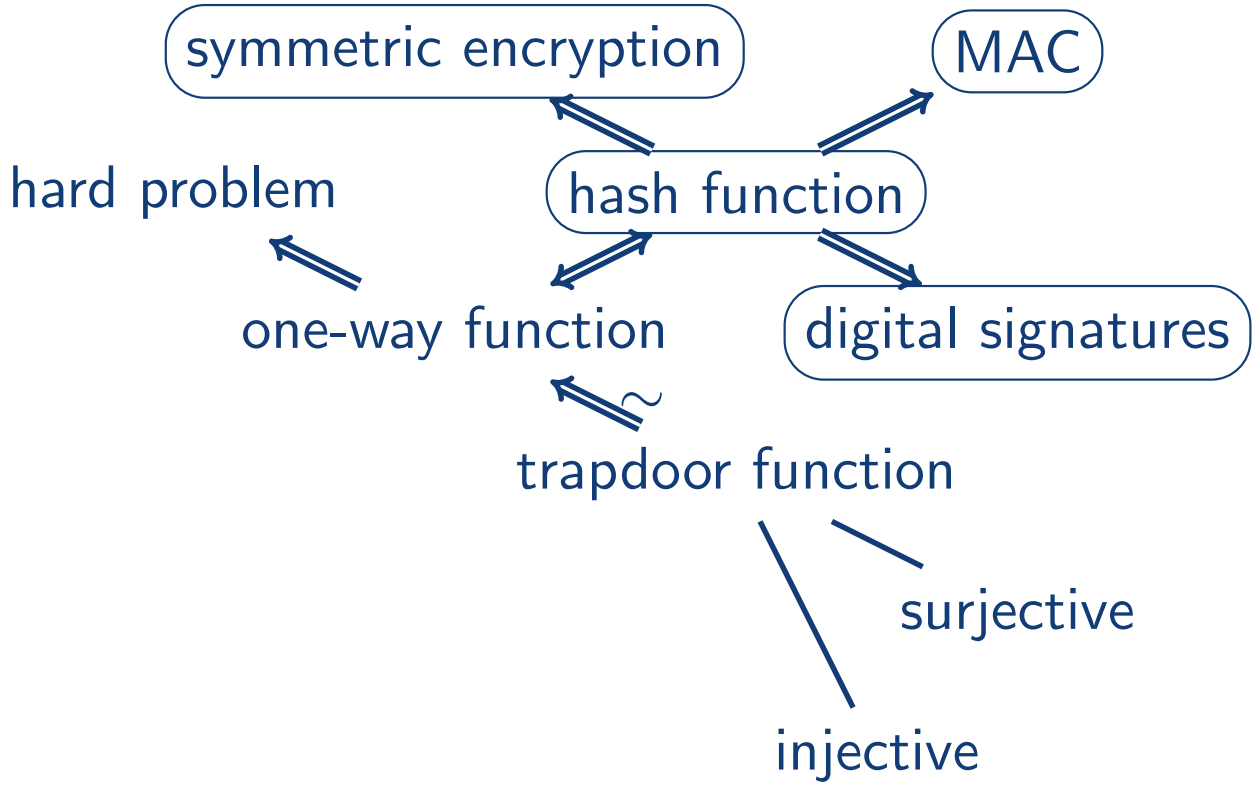

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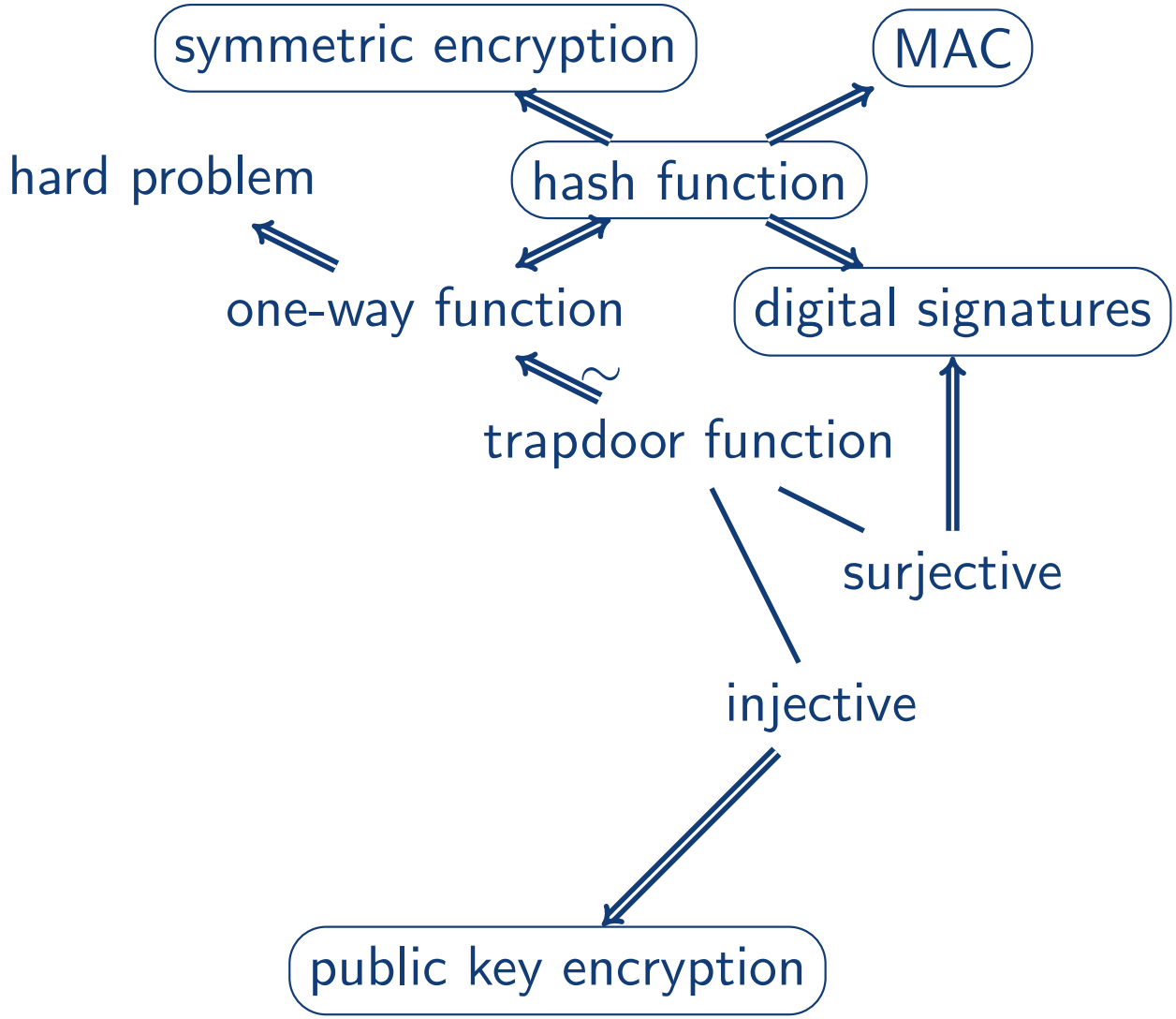

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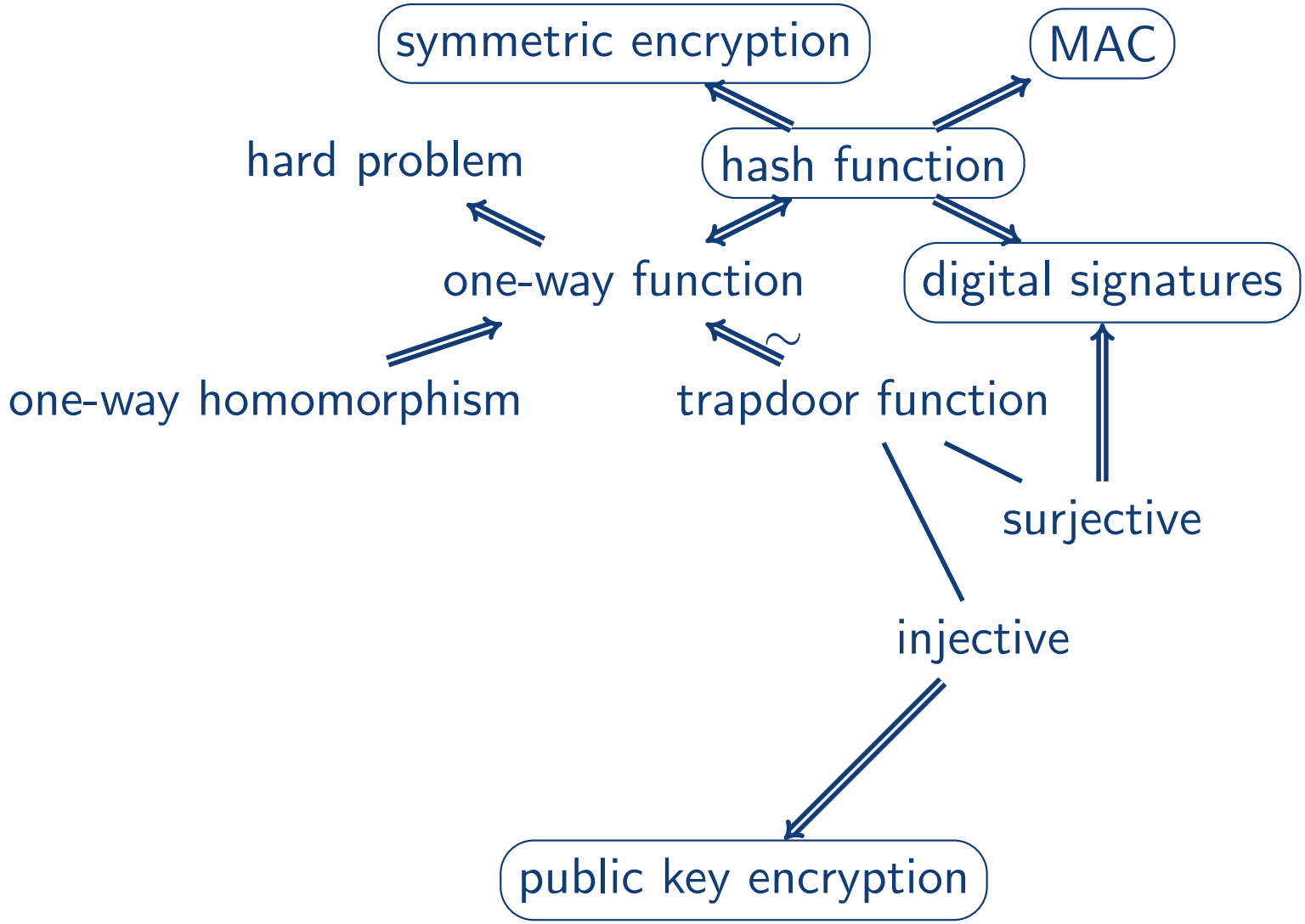

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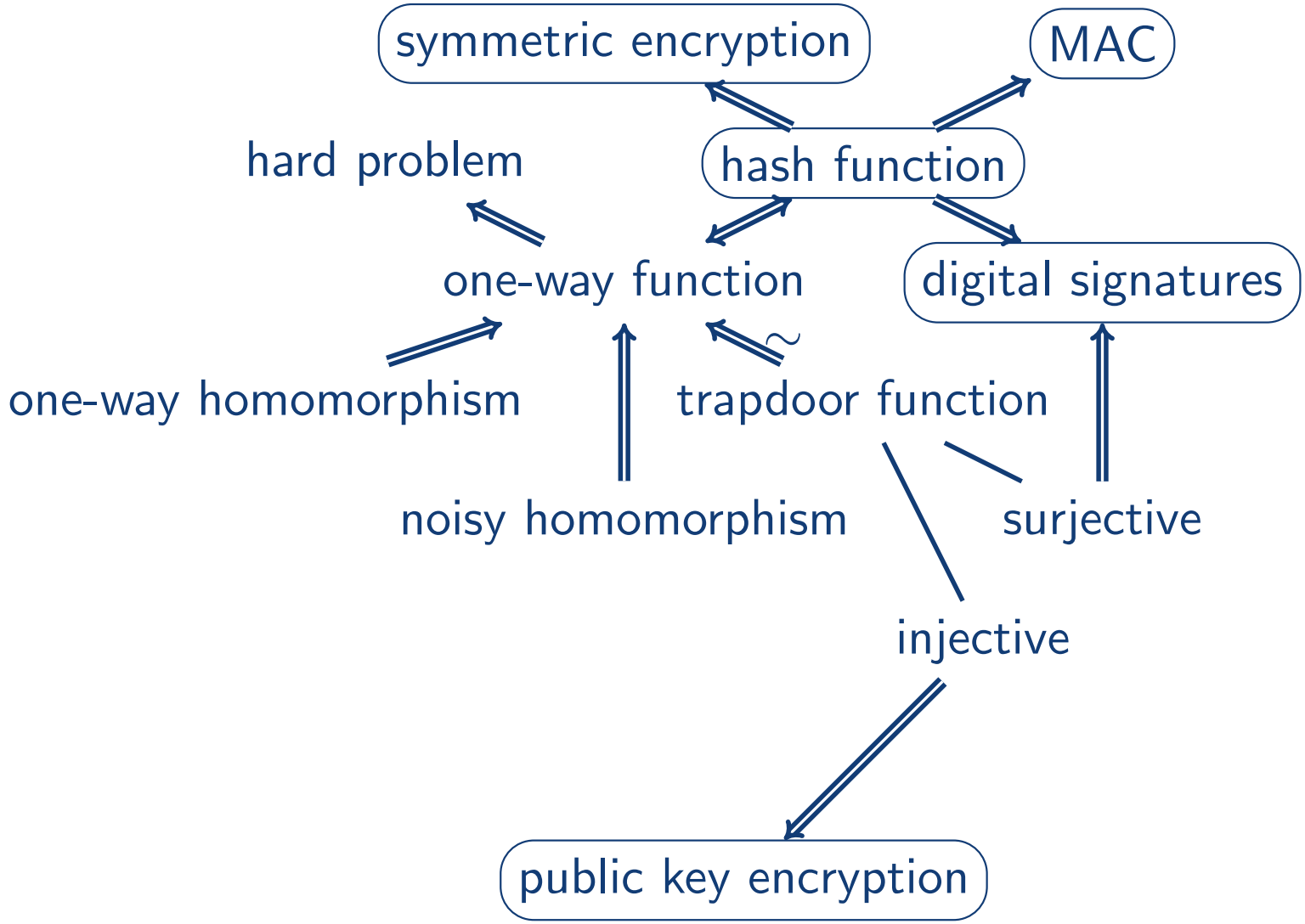

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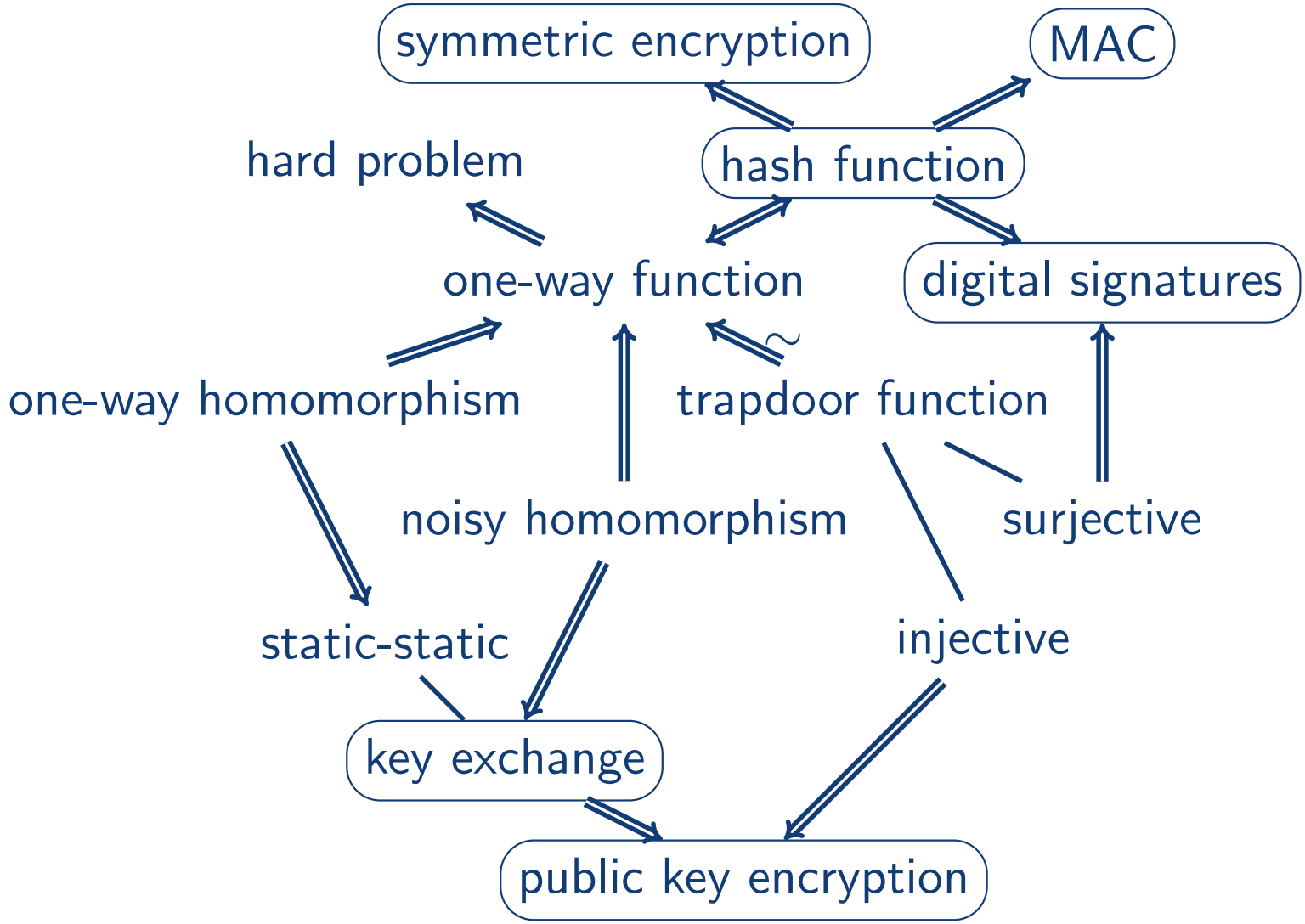

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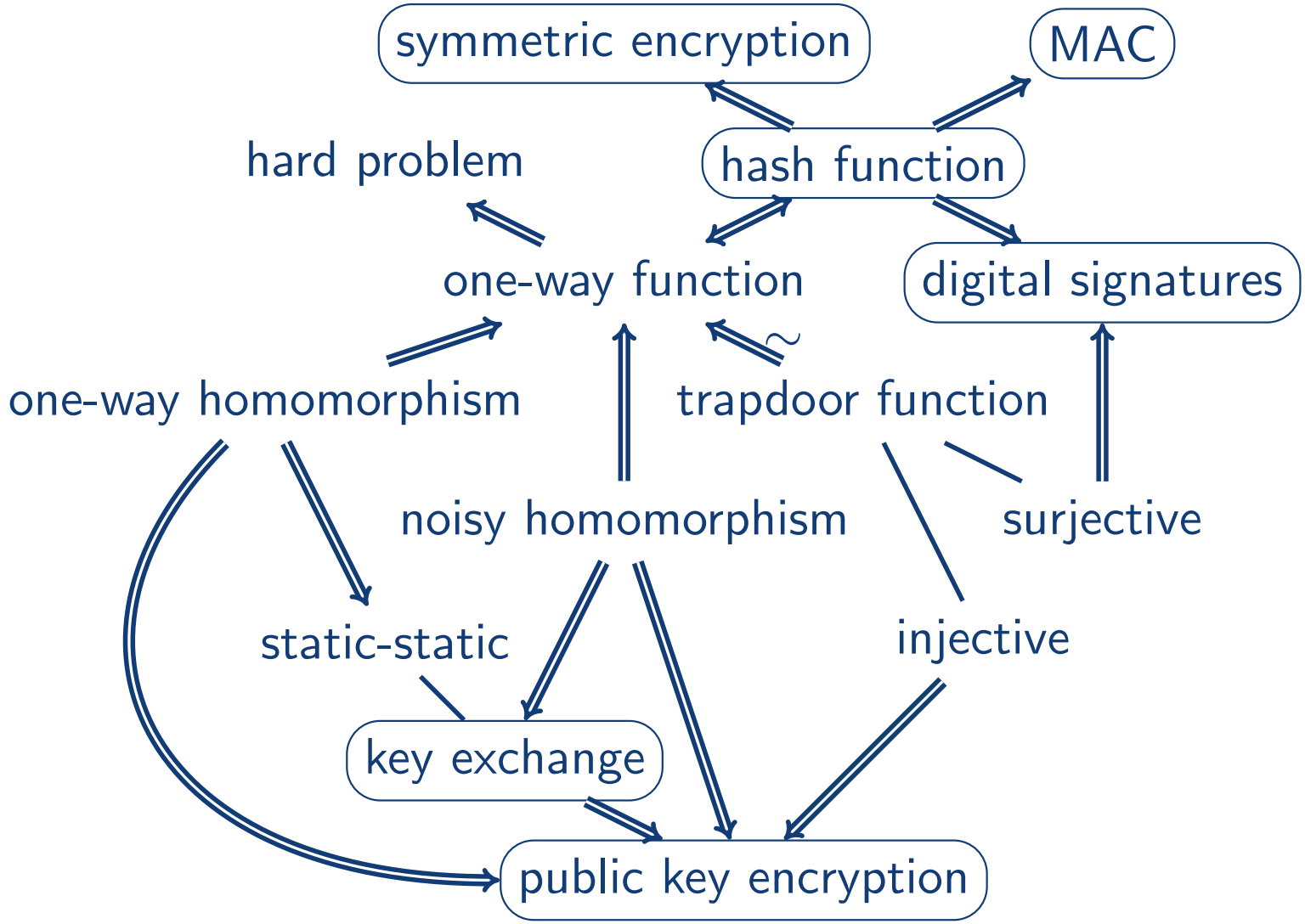

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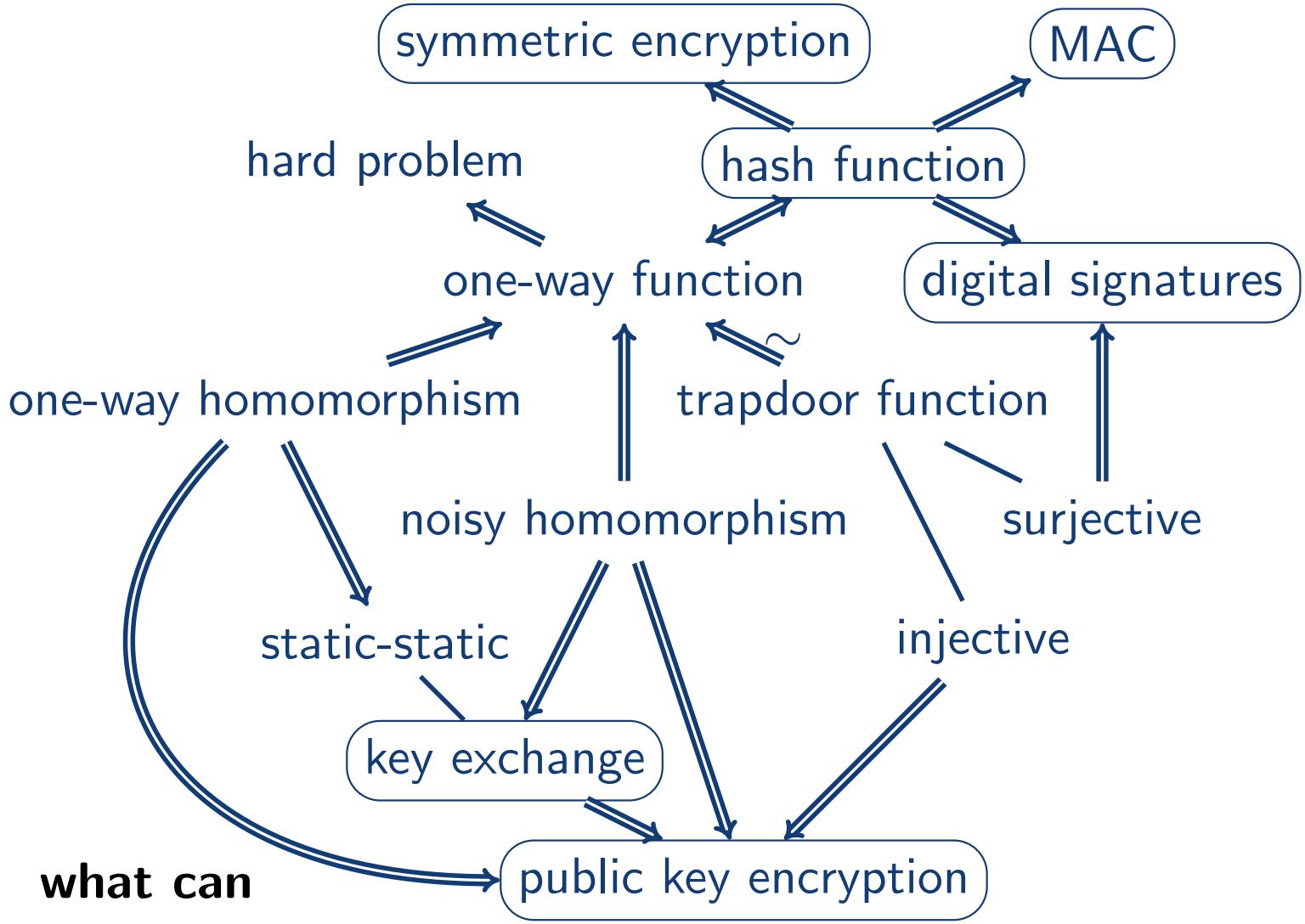


**what can SSNE do?**

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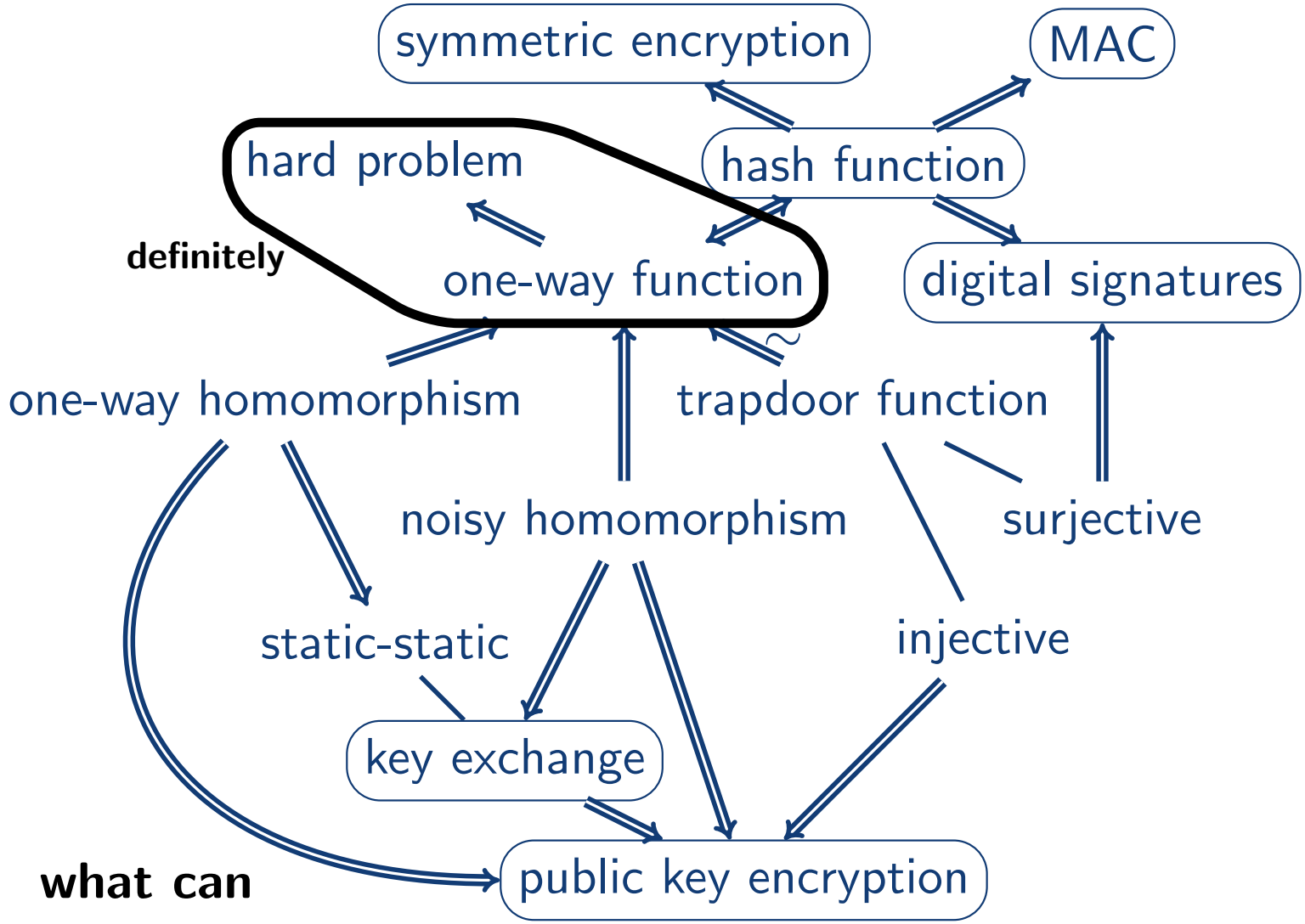

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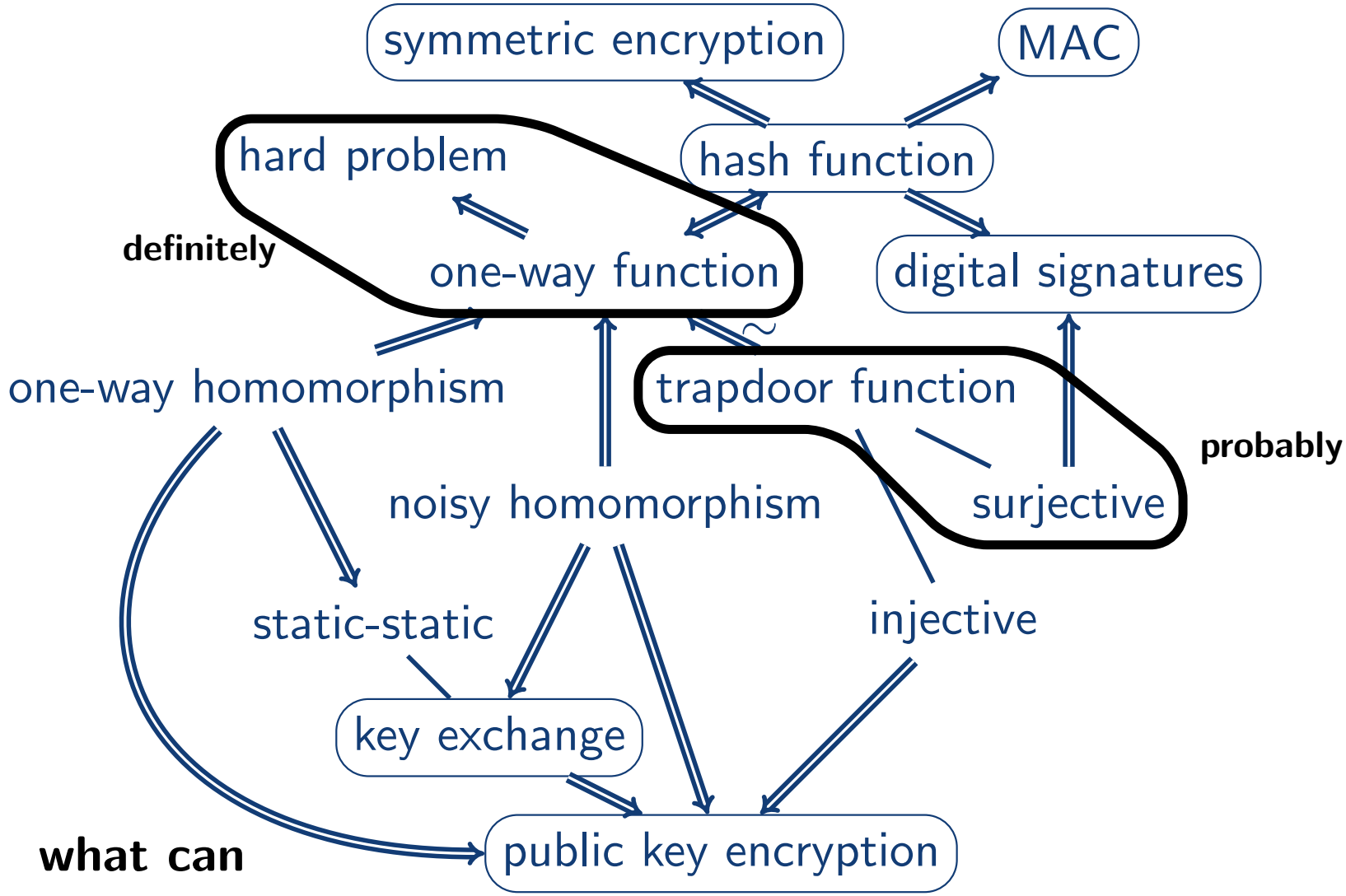

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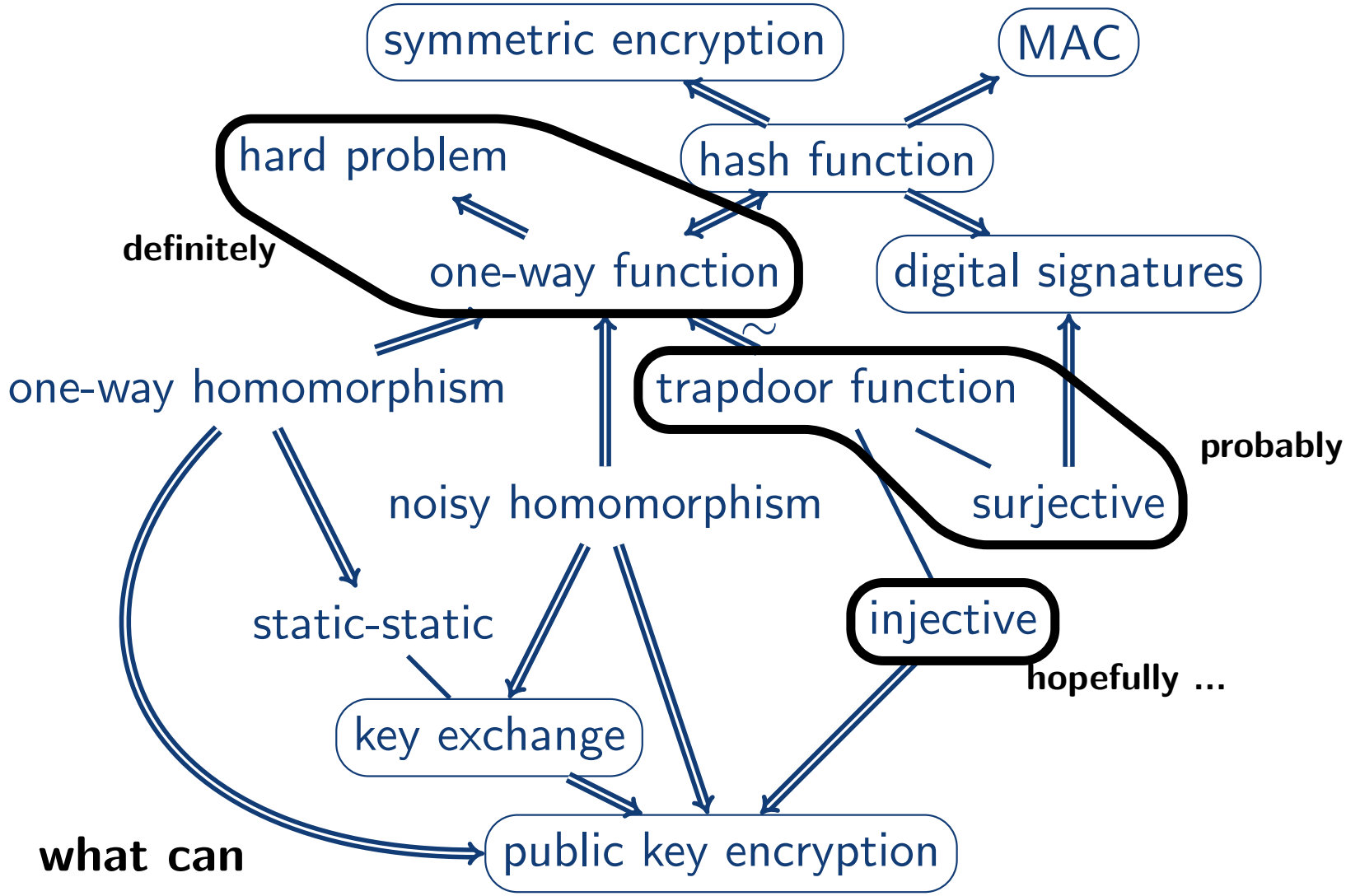

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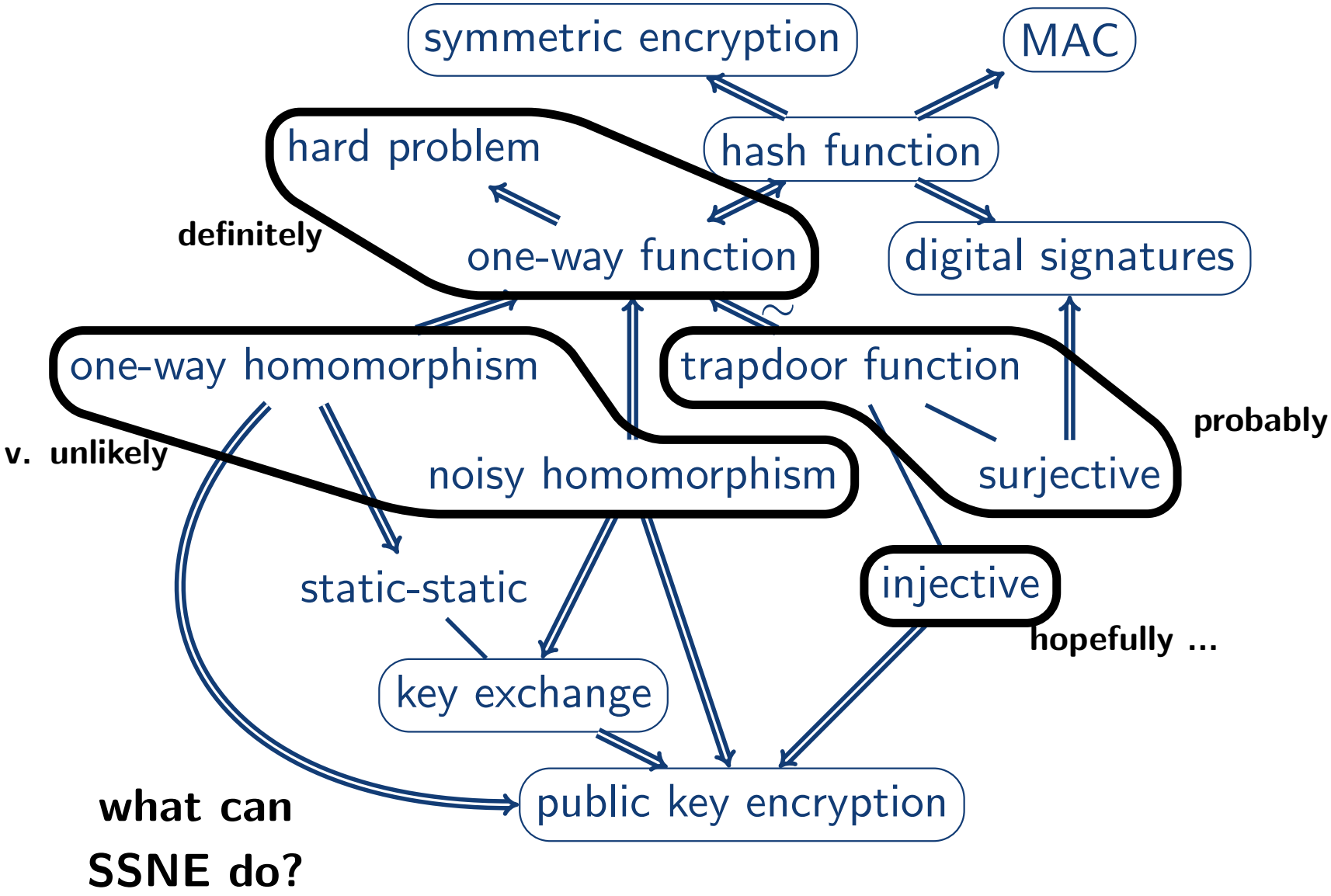


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